Primitive Equations

Weston Anderson

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1 Introduction

These notes are intended to be a brief introduction to the relevant equations of motion used in the atmosphere and ocean. I'm using Vallis (2006) primarily as my reference, but also drawing on notes developed by Ryan Abernathy and Isla Simpson. These were put together for my own benefit (i.e. teaching purposes), but with the idea that they may be helpful to others as well.

1.1 Notation

In what follow I'll assume a Eulerian frame of reference. First a quick bit of notation that I'll use throughout.

vectors will be defined using bold text:

$$\mathbf{x} = (x,y,z)$$

and similarly for vectors of velocities:

$$\mathbf{U} = (u,v,w) = \frac{d\mathbf{x}}{dt}$$

material derivatives:

$$\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + \mathbf{U} \cdot \nabla c$$

1.2 Definition of terms

When introduced to geophysical fluid dynamics for the first time, there can be a lot to keep track of, so below I've compiled a list of common variables used. Feel free to reference this list whenever you can't quite recall how a variable is defined.

$$\Phi = gz \tag{geopotential height}$$

$$b = -g\frac{\rho'}{\rho_0} = -g\frac{T'}{T_0} \tag{buoyancy}$$

$$\phi = \frac{p'}{\rho} \tag{dynamic pressure}$$

$$f \approx f_0 + (\beta y) = f_0 + \left(\frac{\partial f}{\partial y}\right)y = f_0 + \left(\frac{2\Omega cos\theta_0}{a}\right)y \tag{Coriolis parameter}$$

2 Non-rotating equations of motion

2.1 Hydrostatic balance

Critical to our future discussion will be defining the forces that are at play even when a fluid is not moving. So let's begin at the beginning! $\mathbf{F} = m\mathbf{a} = 0$ for a motionless fluid. Excellent, this makes things simple. All we have to do is define the balance of forces which, in this case, are simply the gravitational force balancing the pressure gradient force. If pressure were uniform inside a fluid column, then the pressure gradient force would be 0, but we know this isn't the case in the atmosphere or the ocean. So this leaves us with the following balance:

$$\mathbf{F}_g = \rho \mathbf{g}$$

$$\mathbf{F}_p = -\nabla p$$

$$\mathbf{F}_g - \mathbf{F}_p = 0$$

now we can simplify these if we make a few simple assumptions, such as reducing the vector of gravity to a constant (therefore assuming it doesn't vary much over the domain) and noting that because we've assumed the fluid to be static it is the vertical component of the pressure gradient field that must balance the gravitational force

$$\boxed{\rho g = -\frac{\partial p}{\partial z}} \tag{1}$$

This relation is known as the *hydrostatic balance*. Sear it into your memory because it's important. This equation, however, is only an approximation. We will return to this later.

I should also note here that we can express the hydrostatic equation in terms of *geopotential height* ($\Phi = gz$), which is common in atmospheric sciences. Try to familiarize yourself with both notations.

$$\frac{\partial \Phi}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}$$

2.2 Mass conservation equation

Now we'll begin thinking about motion. First, let's define a simple conservation equation. If a quantity, c, is conserved (as, say, a tracer would be) then we can say that for flow through a given volume in space, the change in concentration of c over time is equal to the advection of c through the given volume:

$$\frac{\partial c}{\partial t} = -\nabla \cdot (\mathbf{U}c)$$

We can include density in this equation to convert it to a mass conservation equation, so that after rearranging we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{U}\rho) = 0$$

This will be one of our primitive equations. But first let's make a few more approximations, the first of which will be to assume that density is constant ($\rho = \rho_0$), as it would be in an incompressible fluid. If we use this approximation in mass conservation, we get:

$$\frac{D\rho_0}{Dt} = \frac{\partial \rho_0}{\partial t} + \nabla \cdot (\mathbf{U}\rho_0) = \frac{\partial \rho_0}{\partial t} + (\mathbf{U} \cdot \nabla \rho_0 + \rho_0 \cdot \nabla \mathbf{U}) = 0$$

and since we have defined density to change in neither time nor space, all of the partial derivatives are zero and we are left with a description of *incompressibility*:

$$\nabla \cdot \mathbf{U} = 0$$

A similar assumption, often made in oceanography, is to divide density into a background value (ρ_0) and an anomaly ($\rho'(x, y, z, t)$), where $\rho' << \rho_0$. We can then return the the conservation of mass as:

$$\nabla \cdot \mathbf{U} = -\frac{1}{\rho_0}\frac{D\rho'}{Dt}$$

And since $\rho' << \rho_0$, we again arrive at an approximation similar to that of incompressibility:

$$\boxed{\nabla \cdot \mathbf{U} \approx 0} \tag{2}$$

2.3 Momentum equation

Let's take what we've learned so far and return to Newton's second law, $\mathbf{F} = m\mathbf{a}$, or for a given volume $\mathbf{F} = \rho\mathbf{a}$. So subbing in the pressure and gravitational forces on the left hand side and the material derivative of velocity on the right ($\mathbf{a} = \frac{D\mathbf{U}}{Dt}$):

$$\rho\frac{D\mathbf{U}}{Dt} = -\frac{\partial p}{\partial z} - \rho g$$

Note that in this equation we have neglected the frictional force and the effect of viscosity, which is often represents the effects of viscosity ($\nu\nabla^2\mathbf{U}$). If we include these, the complete form is:

$$\boxed{\rho\frac{D\mathbf{U}}{Dt} = -\frac{\partial p}{\partial z} - \rho g + \nu\nabla^2\mathbf{U} + Friction} \tag{3}$$

3 The effects of rotation

Up to now we have been considering an inertial reference frame. But in atmospheric and ocean sciences we are often interested in describing motion as observed from the rotating surface of the earth. In this rotating reference frame we need to make adjustments to account for the effects of rotation. First, let us simply describe how a fixed point on the earth (let's call it position vector $\mathbf{r}$) will rotate as viewed from the inertial reference frame:

$$\left(\frac{d\mathbf{r}}{dt}\right)_I = \mathbf{\Omega} \times \mathbf{r}$$

Where $\mathbf{\Omega}$ is the angular velocity of the earth. Now if the point (or, say, air parcel) additionally has a velocity with respect to the surface of the earth, then the position vector is described by:

$$\left(\frac{d\mathbf{r}}{dt}\right)_I = \left(\frac{d\mathbf{r}}{dt}\right)_R + \mathbf{\Omega} \times \mathbf{r} \tag{4}$$

And similarly we can describe the velocity in the rotational frame by:

$$\left(\frac{d\mathbf{U}_R}{dt}\right)_I = \left(\frac{d\mathbf{U}_R}{dt}\right)_R + \mathbf{\Omega} \times \mathbf{r} \tag{5}$$

So that if we now remember that the change in position vector with time is defined as the velocity, we can rewrite equations 4 and 5 as:

$$\mathbf{U}_I = \mathbf{U}_R + \mathbf{\Omega} \times \mathbf{r} \tag{4.1}$$

$$\left(\frac{d\mathbf{U}_R}{dt}\right)_I = \left(\frac{d\mathbf{U}_R}{dt}\right)_R + \mathbf{\Omega} \times \mathbf{r} \tag{5.1}$$

and we can substitute 4.1 into 5.1 to get

$$\left(\frac{d}{dt}(\mathbf{U}_I - \mathbf{\Omega} \times \mathbf{r})\right)_I = \left(\frac{d\mathbf{U}_R}{dt}\right)_R + \mathbf{\Omega} \times \mathbf{r} \tag{5.1}$$

which, after assuming that the rate of rotation is constant and doing a bit of algebra (see Vallis chapter 2.1.2 for this), we end up with:

$$\left(\frac{d\mathbf{U}_R}{dt}\right)_R = \left(\frac{d\mathbf{U}_I}{dt}\right)_I - 2\mathbf{\Omega} \times \mathbf{U}_R - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$$

In this equation, the term on the left hand side is the acceleration in the rotating frame and the first term on the right hand side is the acceleration in the inertial reference frame. The second and third terms (including minus signs) are the *Coriolis force* and the *centrifugal force*, respectively. Neither of these are truly forces, but rather arise due to the rotation of the reference frame. We can note that the centrifugal force acts on the component of $\mathbf{r}$ that is perpendicular to the axis of rotation, and so we can subsume it into the geopotential height term (again, see Vallis 2.1.2 for details). Let's make some quick notes about the Coriolis force though:

$$\mathbf{F}_C = -2\mathbf{\Omega} \times \mathbf{U}_R \tag{6}$$

The Coriolis force plays an important role in atmospheric dynamics, and is often the source of endless confusion for students new to the field. Let's start with two basic properties: (1) The Coriolis force acts only on moving bodies in a rotating frame and (2) because of the rotation of the earth, the Coriolis force deflects moving bodies to the right of their direction of travel in the Northern Hemisphere and to the left in the Southern Hemisphere.

We can combine everything we've learned up to now into the equation of motion in a rotating frame (using the geopotential height notation, $\Phi = gz$):

$$\boxed{\frac{D\mathbf{U}}{Dt} + 2\mathbf{\Omega} \times \mathbf{U} = -\frac{1}{\rho}\nabla p - \nabla\Phi} \quad + (\nu\nabla^2\mathbf{U} + Friction) \tag{7}$$

Note that we've dropped the notation of $\mathbf{U}_R$, since it is now understood that all velocities will be with respect to the rotating reference frame of the earth

4 Approximations

4.1 The f-plane

The first approximation we'll make is the often used "f-plane" approximation. This approximation assumes that locally the spherical surface of the earth may be approximated by a flat plane. This analysis holds so long as the phenomena that we're interested in are of a sufficiently small scale. We'll formalize this criteria later, but for now it's enough to say that this approximation holds approximately for sub-global scale phenomena. To make the f-plane approximation, we will expand the material derivative in 7 but neglect elements of Ω not in the local z direction (again, see Vallis (2006) sect 2.3 for details):

$$\frac{Du}{Dt} - f_0 v = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} + f_0 u = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

where $f_0 = 2\Omega^z = 2\Omega sin\theta_0$ is the coriolis parameter and θ_0 is the latitude on which we are centering our tangent plane. So long as the motion occurs at θ_0, then the f-plane approximation is exactly equivalent to the equation of motion because the local vertical aligns with the rotation vector (Ω).

4.2 The beta-plane

The beta-plane approximation accounts for how the magnitude of the vertical component of the rotation vector changes with latitude (that is, the projection of Ω onto the local vertical (k), see Fig 2.4 from Vallis (2006)).

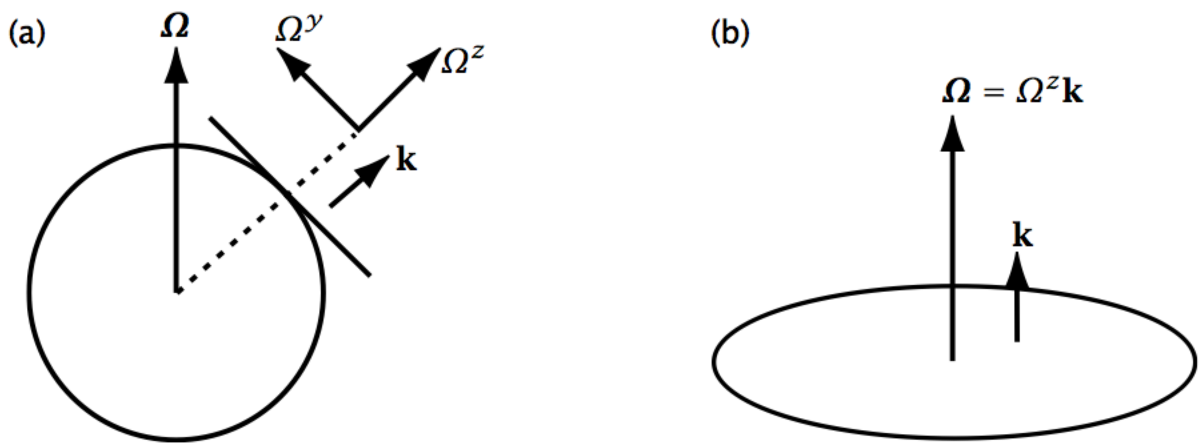

Fig. 2.4 (a) On the sphere the rotation vector $\boldsymbol{\Omega}$ can be decomposed into two components, one in the local vertical and one in the local horizontal, pointing toward the pole. That is, $\boldsymbol{\Omega} = \Omega_y \mathbf{j} + \Omega_z \mathbf{k}$ where $\Omega_y = \Omega \cos\vartheta$ and $\Omega_z = \Omega \sin\vartheta$. In geophysical fluid dynamics, the rotation vector in the local vertical is often the more important component in the horizontal momentum equations. On a rotating disk, (b), the rotation vector $\boldsymbol{\Omega}$ is parallel to the local vertical $\mathbf{k}$.

Figure 1: Figure from Vallis (2006)

We account for changes in the Coriolis parameter with latitude by defining it as:

$$f \approx f_0 + \beta y$$

where $\beta = \partial f/\partial y = (2\Omega cos\theta_0)/a$ given that a is the radius of the earth. This is equivalent to defining the Coriolis parameter as the first two terms in its taylor expansion. The beta-plane approximation accounts for the first order impacts of a rotating sphere on atmospheric/oceanic dynamics, while allowing us to remain in cartesian coordinates.

5 Thermodynamics

Up to this point we have derived the equations of motion in both inertial and rotating reference frames. But to fully describe a fluid parcel, we need also to describe the energy of that parcel. Here we will deal with only the portions of thermodynamics most relevant to fluid dynamics. There are many more relations of interest for atmospheric / oceanic sciences in general, but we will neglect these for the time being. The notation used here follow Borheim and Albrecht (Bohren et al., 2000).

We'll begin with some fundamentals of thermodynamics but since these are not notes on thermodynamics, we will often state (rather than derive) these principles. We begin by describing three important quantities (1) Internal energy, (2) Enthalpy, and (3) Entropy. From these we will use a somewhat modified definition of temperature to build the thermodynamic equation we will use in our primitive equations.

5.1 Internal Energy

So let's first define the total internal energy of a system as the sum of the potential and kinetic energies:

$$U_{int} = K_{int} + P_{int}$$

the change in internal energy with time, therefore, is the rate of heating plus the work:

$$\boxed{\frac{dU_{int}}{dt} = Q + W} \tag{8}$$

This is the first law of thermodynamics! Often we will be considering closed systems in which we define work as resulting from changes in volume. If this is the case, then

$$W = -P\frac{dV}{dt}$$

by definition, and so

$$Q = \frac{dU_{int}}{dt} + P\frac{dV}{dt}$$

5.2 Enthalpy

Let's also define enthalpy (H), which may be thought of as the noun equivalent of the verb "heat". While "heat" is often used as a noun this is not quite correct.

$$H = U_{int} + pV$$

and so the change of enthalpy with time becomes:

$$\frac{dH}{dt} = \frac{dU_{int}}{dt} + (p\frac{dV}{dt} + V\frac{dp}{dt})$$

and from our definition of Q above, we have

$$\boxed{\frac{dH}{dt} = Q + V\frac{dp}{dt}} \tag{9}$$

We can now define a *heat capacity* at constant volume (C_v) and a at a constant pressure (C_p), although we won't explicitly derive either here. The heat capacity at constant volume can be thought of as the change in internal energy with a change in temperature

$$C_v = \frac{\partial U_{int}}{\partial T}$$

We can similarly define a heat capacity at a constant pressure as

$$C_p = \frac{\partial U_{int}}{\partial T}$$

Here we should note that for mixtures of gases both internal energies and enthalpies are additive. This means that, by extension, heat capacities are also additive.

5.3 Entropy and potential temperature

We now define the entropy (S) to be approximately the integral of heating (Q) in a closed system. If we follow the derivation from Bohren et al. (2000)., we arrive at the following relation:

$$\boxed{S = S_0 + C_p ln\left(\frac{T}{T_0}\right) - \mathrm{nk} ln\left(\frac{p}{p_0}\right)} \tag{10}$$

where nk refers to the gas constant.

Now we will define the potential temperature (Θ) as the temperature a parcel of air would have if brought adiabatically and reversibly to a reference height p_0.

$$\Theta = \left(\frac{p_0}{p}\right)^{(\gamma-1)/\gamma} T$$

where $\gamma = \frac{C_p}{C_v}$. And by making use of the ideal gas law ($PV = nkT$) we can do some algebra (see Bohren and Albrecht ch 4.3) to rewrite this as

$$S - S_0 = C_v \gamma ln\left[\left(\frac{T}{T_0}\right)\left(\frac{p_0}{p}\right)^{(\gamma-1)/\gamma}\right] = C_p ln\left(\frac{\Theta}{T_0}\right)$$

And because entropy – which is conserved in adiabatic reversible processes – is proportional to the logarithm of potential temperature, it follows that the potential temperature is also conserved for these processes. This means that potential temperature is conserved following adiabatic motion in an ideal gas.

$$\frac{D\Theta}{Dt} = 0$$

Finally, in the presence of diabatic heating we may use the first law of thermodynamics and the ideal gas law to get the final form of our equation:

$$\boxed{C_p \frac{D\Theta}{Dt} = \frac{\Theta}{T} Q} \tag{11}$$

6 Review

Let's now take everything we've learned and put it together. Because the f-plane approximation is so common, we'll express the equations of motion using this approximation.

$$\rho g = -\frac{\partial p}{\partial z} \qquad \text{(Hydrostatic balance)}$$

$$\nabla \cdot \mathbf{U} = 0 \qquad \text{(conservation of mass)}$$

$$C_p \frac{D\Theta}{Dt} = \frac{\Theta}{T} \qquad \text{(Thermodynamic eq.)}$$

$$\frac{Du}{Dt} - f_0 v = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{(f-plane x-momentum)}$$

$$\frac{Dv}{Dt} + f_0 u = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{(f-plane y-momentum)}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \qquad \text{(f-plane z-momentum)}$$

References

Craig F Bohren, Bruce A Albrecht, and Daniel V Schroeder. Atmospheric thermodynamics. *American Journal of Physics*, 68(12):1159–1160, 2000.

Geoffrey K Vallis. *Atmospheric and oceanic fluid dynamics: fundamentals and large-scale circulation*. Cambridge University Press, 2006.